NOTES · TERM 1 WEEK 1

Integral Calculus (SAMPLE)

Mathematics Extension 1 · Year 12

STUDENT NAME

DATE

PART A

Integration by substitution

KEY IDEA · THE REVERSE CHAIN RULE

Substitution is the integration version of the chain rule. If an integrand has an “inside function” and the derivative of said “inside function” also appears, we can replace the inside function by a new variable like so:

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du, \qquad u = g(x), \quad du = g'(x)\,dx.$$

A reliable process is:

1. Find an appropriate choice for u, usually the the inner expression;
2. Differentiate to find du;
3. Rewrite the integral in terms of u;
4. Integrate with respect to u;
5. Substitute back in terms of x and add C.

The substitution is useful only if we are able to remove the original variable. If an unwanted x remains, reconsider the choice of u or rewrite the integrand using $x = g^{-1}(u)$.

PART B

Integration by substitution — examples

WORKED EXAMPLE · A POWER OF A QUADRATIC

Find

$$\int x(1+x^2)^5\,dx.$$

The repeated expression is $1+x^2$, and its derivative $2x$ is already present up to a constant. Let

$$u = 1+x^2, \qquad du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

Therefore

$$\begin{aligned}\int x(1+x^2)^5\,dx &= \frac{1}{2}\int u^5\,du \\ &= \frac{u^6}{12} + C \\ &= \boxed{\frac{(1+x^2)^6}{12} + C}.\end{aligned}$$

A quick check is to differentiate the answer: the factor of 12 disappears and the chain rule supplies the missing $2x$.

WORKED EXAMPLE · A LOGARITHMIC FORM

Find

$$\int \frac{3x^2}{1+x^3}\,dx.$$

Choose the denominator:

$$u = 1+x^3, \qquad du = 3x^2\,dx.$$

Then

$$\int \frac{3x^2}{1+x^3}\,dx = \int \frac{1}{u}\,du = \boxed{\ln|1+x^3| + C}.$$

Recognise the pattern $g'(x)/g(x)$.

WORKED EXAMPLE · A ROOT WITH A HIGHER-DEGREE INNER FUNCTION

Find

$$\int x^2\sqrt{x^3+4}\,dx.$$

Let $u = x^3 + 4$. Then $du = 3x^2\,dx$, so

$$\begin{aligned}\int x^2\sqrt{x^3+4}\,dx &= \frac{1}{3}\int u^{1/2}\,du \\ &= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C \\ &= \boxed{\frac{2}{9}(x^3+4)^{3/2} + C}.\end{aligned}$$

PART C

Practice Questions

Question 1

Find

$$\int 6x(3x^2+1)^4\,dx.$$

Question 2

Find

$$\int \frac{x^2}{x^3+5}\,dx.$$

Question 3

Find

$$\int \frac{e^{2x}}{1+e^{2x}}\,dx.$$

Question 4

Find

$$\int \frac{x}{\sqrt{9-x^2}}\,dx.$$

Question 5

Evaluate

$$\int_0^{\pi/4} \tan x \sec^2 x\,dx.$$

Question 6

Find

$$\int \frac{\cos x}{(2+\sin x)^3}\,dx.$$

Question 7

Find

$$\int x^3\sqrt{1+x^4}\,dx.$$

Question 8

Find

$$\int \frac{x^2-1}{x^4+3x^2+1}\,dx.$$

KEY IDEA

Beyond its use for simplifying expressions of the form $\int f(g(x))g'(x)dx$, u-substitutions can be used to reduce the amount of calculations required. Consider the integral

$$\int x(x+6)^7 dx.$$

It is perfectly possible to integrate by expanding the power, but this is of course inefficient. Instead, we apply the substitution:

$$u = (x+6), \qquad du = dx, \text{ noting } x = u-6,$$

$$\begin{aligned}\int (u-6)u^7 du &= \frac{u^9}{9} - \frac{3u^8}{4} + C \\ &= \frac{(x+6)^9}{9} - \frac{3(x+6)^8}{4} + C\end{aligned}$$

PART D

Practice Questions

Question 9

Find

$$\int \frac{2x}{(\sqrt{x}-1)^5} dx$$

Question 10

Find

$$\int \frac{2x+7}{(x-12)^4} dx$$

PART E

Sketching a primitive

KEY IDEA

A primitive F of f satisfies $F'(x) = f(x)$. The graph of f therefore controls the *gradient* of F.

- $f(x) > 0 \Rightarrow F$ is increasing.
- $f(x) < 0 \Rightarrow F$ is decreasing.
- $f(x) = 0 \Rightarrow F$ is stationary.
- A change $f : - \to +$ gives a local minimum of F; a change $f : + \to -$ gives a local maximum.
- Large $|f(x)|$ means F is steep; small $|f(x)|$ means F is flat.

Because $F''(x) = f'(x)$, the *slope* of f determines the concavity of F:

- f increasing $\Rightarrow$ F is concave up;
- f decreasing $\Rightarrow$ F is concave down;
- a local extremum of f usually corresponds to a point of inflection of F.

KEY IDEA

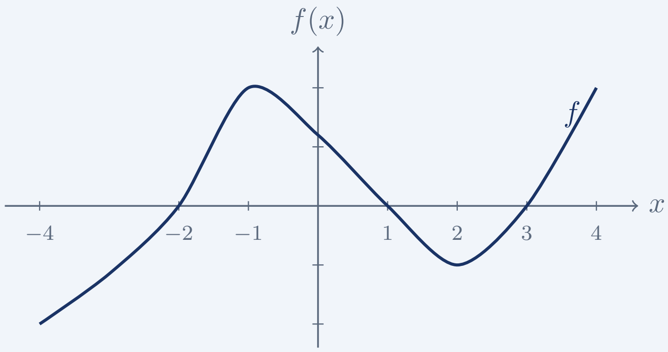
Knowing $F' = f$ determines the *shape* of F, but not its vertical position: all primitives differ by a constant and there are hence an infinite number of possible curves. An initial condition such as $F(a) = b$ fixes the vertical shift, allowing us to draw a specific curve.

PART F

Sketching a primitive – example

WORKED EXAMPLE · FROM A DERIVATIVE GRAPH TO A PRIMITIVE

The graph below represents $y = f(x)$, where $F'(x) = f(x)$. Sketch one possible primitive F satisfying $F(0) = 1$.



- F decreases for $x < -2$, increases on $(-2, 1)$, decreases on $(1, 3)$, and increases for $x > 3$.
- Therefore F has local minima at $x = -2$ and $x = 3$, and a local maximum at $x = 1$.
- Since f has a local maximum near $x = -1$ and a local minimum near $x = 2$, the concavity of F changes near those points.
- The condition $F(0) = 1$ fixes the vertical position of the sketch.

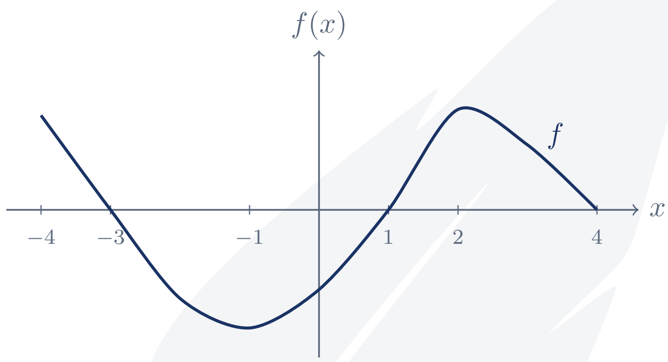
Hence, we can sketch the primitive:

PART G

Practice Questions

Question 11

The graph below is $y = f(x)$ and $F'(x) = f(x)$.



(a) State the intervals on which F is increasing and decreasing.

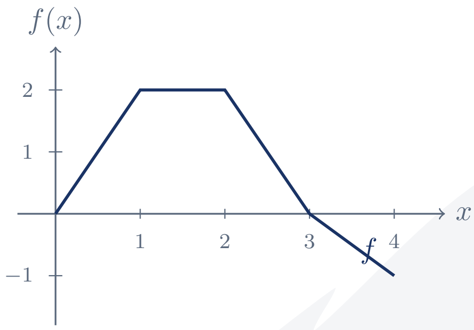
(b) Identify and classify the stationary points of F that are visible from the graph.

(c) State where F is concave up and concave down, as accurately as the graph allows.

(d) Sketch a possible graph of F satisfying $F(0) = 0$.

Question 12

Let f be the piecewise-linear function shown below.



(a) Let G be the primitive of f, with $G(0) = 0$. Sketch G on $0 \leq x \leq 4$. Your sketch should show the correct values at $x = 0, 1, 2, 3$ and the correct behaviour at $x = 3$.

END OF WORKSHEET